Outline

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2 Young-Laplace Eqn

3 Deriving a Boundary Condition

4 Computing the solutions and eigenfrequencies

5 Closing Remarks

My Goal

- To model the waves which form on the surface of a water balloon impinging on a surface
    - Look at acoustic (pressure) waves created within the water balloon
    - Look at waves formed from deformation of the balloon surface

Figure : Waves formed on a water balloon surface

Update

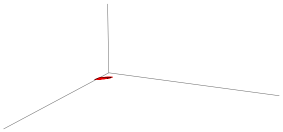


Figure : A travelling Gaussian isobar impinging from below a membrane

- Previous approach looked at an acoustic driving force driving oscillations on a membrane
- This is mathematically complicated: two coupled PDEs (the acoustic pressure wave, and the wave equation on the surface)
- Better approach: try modelling the surface force as the surface tension of a non-wetting droplet
- This is governed by the Young-Laplace Equation

Brief Review

Fluid mechanics: describe the velocity of \elements" of the uid, $\vec{u}$

If irrotational ow: $\nabla \times \vec{u} = 0$, therefore $\vec{u} = \nabla$

is called the **velocity potential** and it satis es Laplace's Equation

$$\nabla^2 \quad = 0$$

Goal: Solve the Laplace equation for the a droplet.

- Velocity potential of uid at surface of balloon will give velocity of balloon surface
- Need a boundary condition to solve the Laplace Equation

Young-Laplace Equation

The Young-Laplace Equation describes the pressure di erence at the surface between two uid media:

$$p =$$

- $p = p_1 - p_2$ where p_1 is pressure in medium 1 and p_2 is pressure in medium 2
- is the surface tension (units J/m^2 or N/m)
-

A Slightly Deformed Sphere

Need to calculate the curvature of a sphere that is slightly deformed

Consider radius of slightly deformed sphere to be

$$r(\; ; \;) = R + \quad (\; ; \;)$$

- R is the original radius
- is a small deviation from R

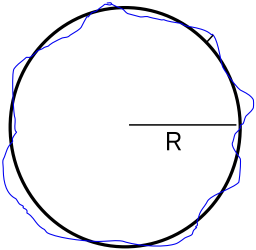


Figure : Near-sphere, with slight changes in radius

What is $\frac{1}{R_1} + \frac{1}{R_2}$?

Can be calculated by equating the infinitesimal change in the surface area

$$A = \iint \quad \frac{1}{R_1} + \frac{1}{R_2} \quad dA$$

– small change in radius.
Alternatively, calculating the surface area of the deformed sphere:

$$A = \iint \quad (R + \quad) \sqrt{1 + \nabla^2 r} \quad dA$$

which for small change becomes

$$A = \iint \quad \frac{2}{R} - \frac{2}{R^2} - \frac{1}{R^2} \quad \frac{1}{\sin^2} \frac{\partial^2}{\partial \quad ^2} + \frac{1}{\sin} \frac{\partial}{\partial} \quad \sin \frac{\partial}{\partial} \quad dA$$

equating the integrands we get...

Surface Pressure and Fluid Pressure

Young-Laplace Equation becomes

$$p = p_f - p_{air} = \frac{2}{R} - \frac{2}{R^2} - \frac{1}{R^2} \quad \frac{1}{\sin}\frac{\partial}{\partial} \quad \sin \frac{\partial}{\partial} \quad + \frac{1}{\sin^2}\frac{\partial^2}{\partial^2}$$

- p_{air} is constant, ambient
- $p_f = - \frac{\partial}{\partial t}$

At the surface $\partial \;/\partial t = \partial \;/\partial r$. Differentiate the above w.r.t. time and substitute:

The boundary condition

$$\frac{\partial^2}{\partial t^2} - \frac{}{R^2} \quad 2\frac{\partial}{\partial r} + \frac{\partial}{\partial r} \quad \frac{1}{\sin}\frac{\partial}{\partial} \quad \sin \frac{\partial}{\partial} \quad + \frac{1}{\sin^2}\frac{\partial^2}{\partial^2} \quad = 0$$

Contact Pressure

The pressure on the surface isn't p_{air} at every point of the sphere. At the bottom there is a Dirac delta pressure

$$P_f = \quad (r = R; \quad = \quad ; \quad = 0)$$

this changes the boundary condition equation (adds an extra term)

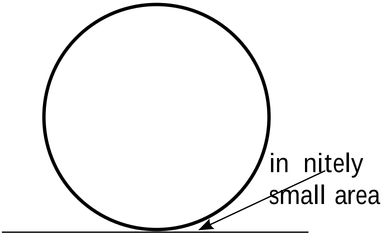


Figure : A sphere droplet resting on a plane

Solution of Laplace's Equation

Look for a solution

$$= \exp(-i\,!\,t)f(r;\ \ ;\ \ )$$

so

$$\begin{aligned} \nabla^2 \quad &= 0 \\ \nabla^2(\exp(-i\,!\,t)f(r;\ \ ;\ \ )) &= \\ \exp(-i\,!\,t)\nabla^2 f(r;\ \ ;\ \ ) &= \\ \nabla^2 f(r;\ \ ;\ \ ) &= 0 \end{aligned}$$

so f must solve Laplace's Equation.

Oscillations of a Water Balloon

Sven Isaacson

Background

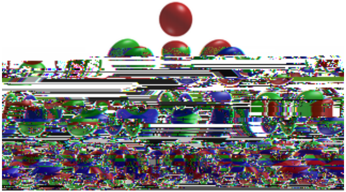
Young-Laplace Eqn

Deriving a Boundary Condition

Computing the solutions and eigenfrequencies

Closing Remarks

Spherical Harmonics



Plugging in our solution

The boundary condition

$$\frac{\partial^2}{\partial t^2} - \frac{}{R^2} \quad 2\frac{\partial}{\partial r} + \frac{\partial}{\partial r} \quad \frac{1}{\sin}\frac{\partial}{\partial} \quad \sin \quad \frac{\partial}{\partial} \quad + \frac{1}{\sin^2}\frac{\partial^2}{\partial \quad ^2} \quad = 0$$

with

$$= \exp(-i\omega t) r^l Y_{l,m}(\, ; \,)$$

reduces to

$$\omega_l^2 = \frac{l(l-1)(l+2)}{R^3}$$

or, when the expansion of the contact force is included

$$\omega_l^2 = \frac{}{R^3}\frac{l(l-1)(l+2)}{1+\sqrt{\overline{\quad}}\,(2l+1)/4}$$

Summary

- Surface e ects should be treated as surface tensions, to avoid two coupled PDEs
- Young-Laplace equation governs pressure di erences caused by surface tension
- The Y-L equation can be used to get a boundary condition of the Laplace equation for uid velocity potential

Conclusions

There are some problems with this model

- Applied pressure is not just at a point, but grows with time
- Di cult to determine \surface tension" of a balloon { wouldn't expect this to be equal to the elastic tension
- This is theory is for *small* droplets for which gravity is negligible to capillary action

However, this my best attempt yet

- Neatly ties together the surface term and the internal velocity eld
- Reduces to the easily solved Laplace equation, for the velocity potential

Future Work

- Account for gravity waves in the water balloon
- Treat contact force as an expanding area as a function of time, rather than point
- Compare measured values to predicted

Acknowledgements and References

Thanks to:

- Dr. Voytas, for advising my research
- Dr. Sancier-Barbosa for help through out with some of the PDEs (that I didn't end up using)
- Dr. Flesich, for providing feedback on my dry run

References:

- Courty, *Oscillating droplets by decomposition on the spherical harmonic basis*